The Describing Function: A classical nonlinear control technique of great practical interest

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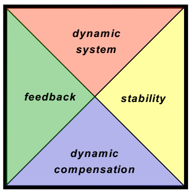

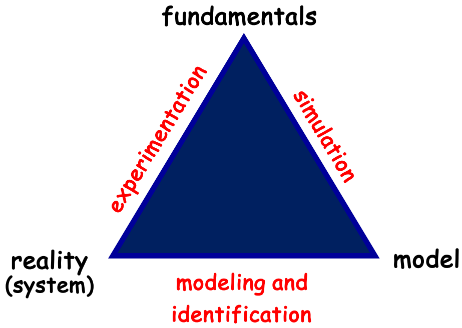

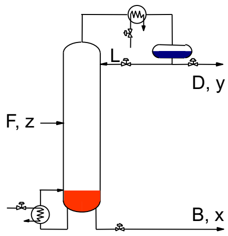

D, Muñoz de la Peña, M. Domínguez, F. Gómez-Estern, O. Reinoso, F. Torres, S. Dormido. "Estado del arte de la educación en automática", *Revista Iberoamericana de Automática e Informática Industrial*, 19, 2022, 117-131; doi: 10.4995/riai.2022.16989

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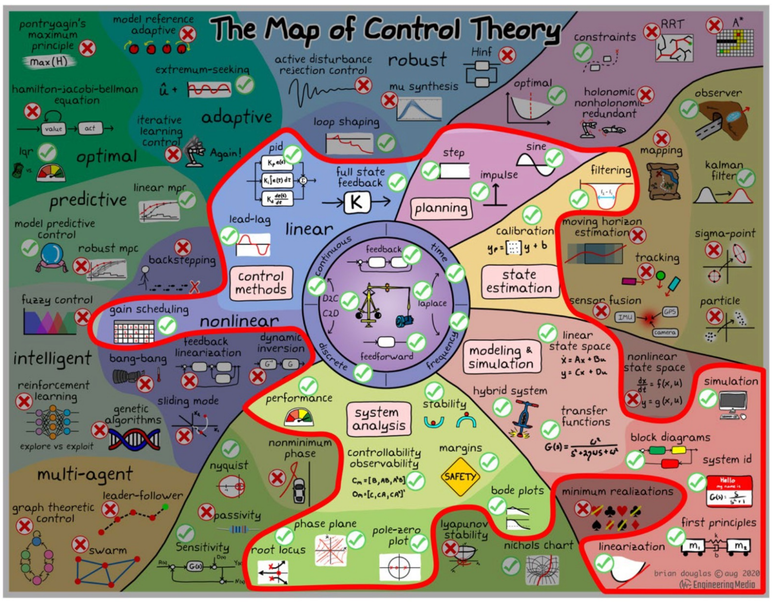
1. Introduction



Brian Douglas https://engineeringmedia.com/

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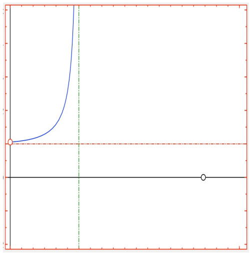
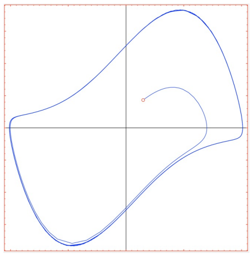
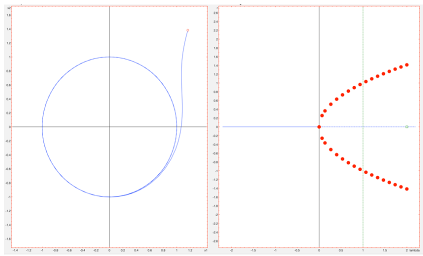
1. Introduction



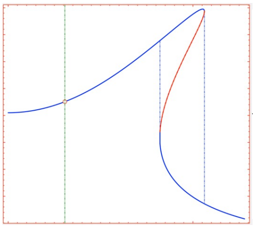
Finite escape time



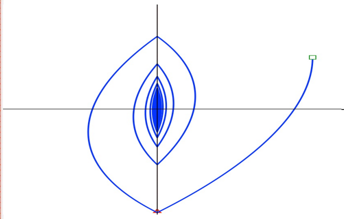
Limit cycle



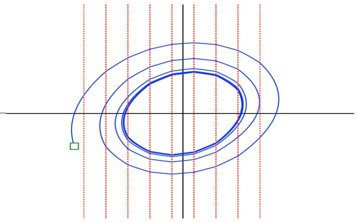
Hopf bifurcation



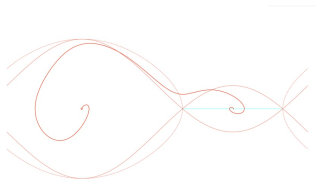
Jump resonance



Zenon effect and sliding mode



Multiple limit cycles



Inverted pendulum

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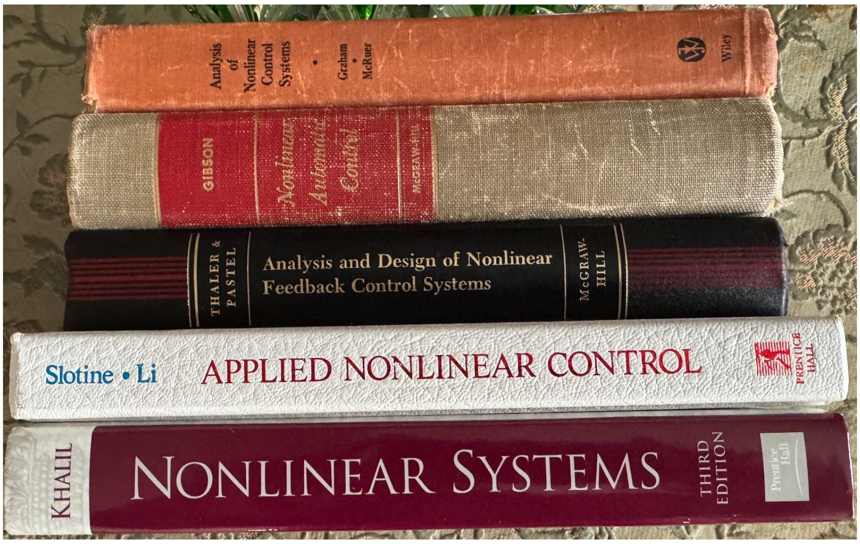
1. Introduction



1961

1962

1963

1991

2002

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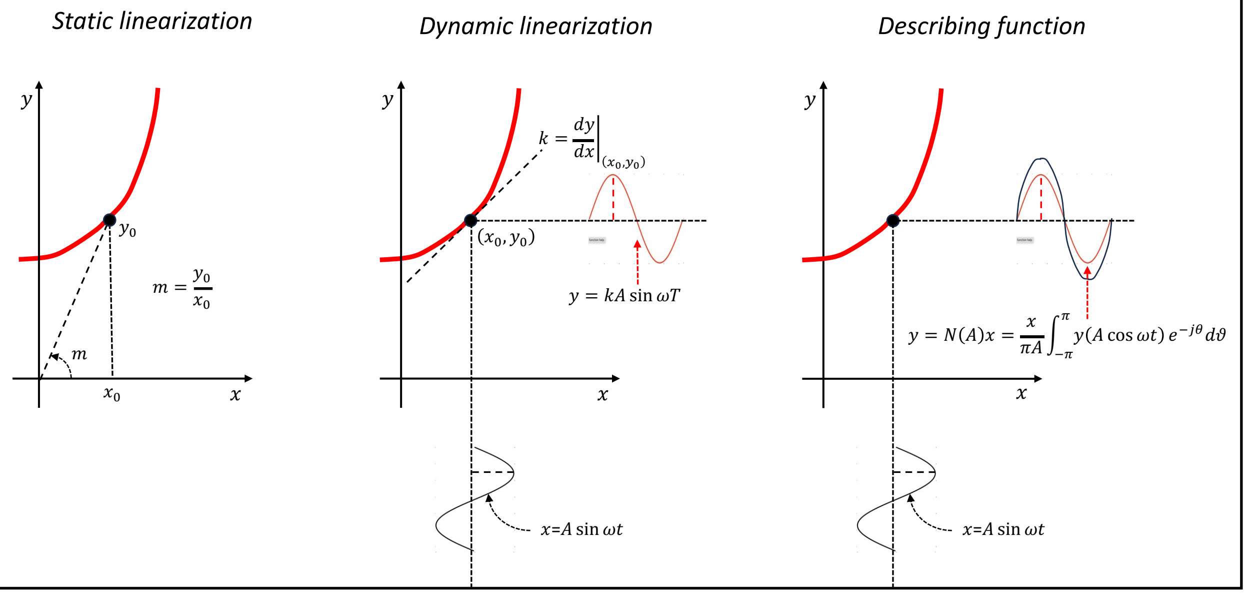
2. Describing function fundamentals

The idea of linearization



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2. Describing function fundamentals

Some historical notes

- During the years 1947 to 1950, in at least five different countries a new point of view came into being for the determination of *limit cycles.*
- This new point of view was essentially a physically motivated harmonic linearization or harmonic balance (Krylov and Bogoliuvov).
- Since then a veritable wealth of material has been published related to the DF and modifications.
- The fact that DF is an approximate method has led to a certain skepticism on the part of control theorists concerned with ensuring the existence or not of limit cycles.

Kochenburger, R. J.: A Frequency Response Method for Analyzing and Synthesizing Contactor Servomechanisms, *Transactions AIEE*, 11, 69, 1950, 270-284.

Goldfarb, L. C.: On Some Nonlinear Phenomena in Regulatory Systems, *Automatika i Telemekhanika*, 8, 5, 1947, 349-383. Translation: R. Oldenburger (ed.), "Frequency Response," The Macmillan Company, New York, 1956, pp. 239-259

Tustin, A.: The Effects of Backlash and of Speed-dependent Friction on the Stability of Closed-Cycle Control Systems, *J. IEE*, IIa, 94, 1947, 143-151.

Oppelt, W.: Locus Curve method for Regulators with Friction, *Z. Deut. Ingr,,* Berlin, 90, 1948, 179-183. Translated in Report 1691, National Bureau of Standards, Washington, 1952

Dutilh, J.: Théorie des servomecanismes à relais, *L'Onde électrique*, 30, 1950, 438-445

R. W. Bass. "Mathematical Legitimacy of Equivalent Linearization by Describing Functions", *First IFAC World Congress,* Moscow, 1960

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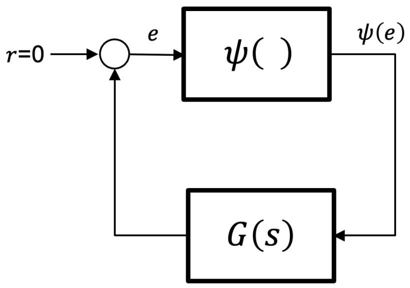
2. Describing function fundamentals

Introduction to the harmonic balance method



$G(s)$ is a strictly proper, rational transfer function

Ψ is a time-invariant, memoryless nonlinearity

$$(G\psi + 1)e = 0 \qquad (1)$$

A periodic solution satisfies $e\left(t + \frac{2\pi}{\omega}\right) = e(t) \quad \forall t$

The idea of the method is to represent a periodic solution by a Fourier series and seek a frequency ω and a set of Fourier coefficients that satisfy (1).

$$e(t) = \sum_{k=-\infty}^{\infty} a_k e^{(jk\omega t)} \quad (2) \quad \Longrightarrow \quad \psi\big(e(t)\big) = \sum_{k=-\infty}^{\infty} c_k e^{(jk\omega t)} \quad (3)$$

Substituting (2) and (3) into (1) $\Longrightarrow$

$a_k = a_k$ where each complex coefficient c_k is a function of all a_i's

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2. Describing function fundamentals

Introduction to the harmonic balance method

$G(jk\omega)c_k + a_k = 0 \;\; \forall k$ because $G(jk\omega) = \bar{G}(-jk\omega),\, a_k = \bar{a}_{-k},$ y $c_k = \bar{c}_{-k}$ $\Longrightarrow$ $G(jk\omega)c_k + a_k = 0 \;\; k \geq 0 \;\; (4)$

it is an infinite-dimensional equation, which it can hardly solve

$\lim_{\omega \to \infty} G(j\omega) \longrightarrow 0 \;\; \Longrightarrow \;\; \exists\, q > 0 \,|\, \forall\, k > q \;\; |G(jk\omega)|$ is small enough to replace $G(jk\omega)$ and a_k by 0

$G(jk\omega)\hat{c}_k + \hat{a}_k = 0 \quad k = 0,1,2,\cdots,q$ that it is a finite-dimensional problem

Classical describing function: $\hat{a}_k = 0 \quad k > 1 \quad \Longrightarrow \quad G(s)$ must have sharp "low-pass filtering

$G(0)\hat{c}_0(\hat{a}_0, \hat{a}_1) + \hat{a}_0 = 0$ → one real equation — two real unknowns $\omega\ y\ \hat{a}_0$

$G(j\omega)\hat{c}_1(\hat{a}_0, \hat{a}_1) + \hat{a}_1 = 0$ → one complex equation — one complex unknown $\hat{a}_1$

the time origin is arbitrary for an autonomous system $\Longrightarrow$ if $(\hat{a}_0, \hat{a}_1)$ satisfies the equation $(\hat{a}_0, \hat{a}_1 e^{j\theta})$ will give another solution

to take care of this no uniqueness, the first harmonic of $e(t)$ is taken as: $a \sin \omega t, a \geq 0$

The time origin is chosen such that the phase of the first harmonic is zero

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2. Describing function fundamentals

Introduction to the harmonic balance method

$$G(0)\hat{c}_0(\hat{a}_0, \hat{a}_1) + \hat{a}_0 = 0 \quad (1)$$
$$G(j\omega)\hat{c}_1(\hat{a}_0, \hat{a}_1) + \hat{a}_1 = 0 \quad (2)$$

where $\hat{a}_1$, may be taken to be real

$(1) \implies \hat{a}_0 = \hat{a}_0(\hat{a}_1)$, if $G(0) = 0$, then $\hat{a}_0 = 0$

and if ψ is an odd function $(\psi(-e) = -\psi(e))$ then $\hat{c}_0 = \hat{a}_0 = 0$ is always a posible solution

Define the describing function N of the nonlinearity ψ by:

$$N(\hat{a}_1) = \frac{\hat{c}_1(\hat{a}_0(\hat{a}_1), \hat{a}_1)}{\hat{a}_1} \quad (3)$$

$\implies$ equation (2) becomes: $(G(j\omega)N(\hat{a}_1) + 1)\,\hat{a}_1 = 0 \quad (4)$

Since we are not interested in a solution with $\hat{a}_1 = 0$, then (4) $\implies$ $G(j\omega) = -\dfrac{1}{N(\hat{a}_1)}$

The describing function N is obtained by applying a sinusoidal signal $a \sin \omega t$ at the input of ψ and by calculating the ratio of the Fourier coefficient of the first harmonic at the output to a

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2. Describing function fundamentals

$$G(j\omega) = -\frac{1}{N(A)} \quad (1)$$

The describing function method states that if (1) has a solution (A_0, ω_0), then there is "probably" a periodic solution of the system with frequency and amplitude (at the input of the nonlinearity) close to A_0 and ω_0.

Conversely, if (1) has no solutions, then the system "probably" does not have a periodic solution.

More analysis is needed to replace the word "probably" with "certainly" and to quantify the phrase "close to A_0 and ω_0" when there is a periodic solution.

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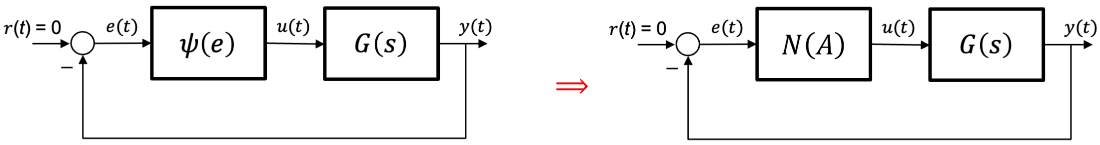
2. Describing function fundamentals

$$G(j\omega) = -\frac{1}{N(A)}$$



Characteristic equation: $1 + N(A)G(j\omega) = 0$

$$N(A) = \frac{j}{\pi A}\int_0^{2\pi} u(A\sin\psi)\, e^{-j\psi}\, d\psi = n_p(A) + j n_q(A)$$

$$n_p(A) = \frac{1}{\pi A}\int_0^{2\pi} u(A\sin\psi)\sin\psi\, d\psi$$

$$n_q(A) = \frac{1}{\pi A}\int_0^{2\pi} u(A\sin\psi)\cos\psi\, d\psi \qquad n_q(A) = 0$$

(if the nonlinearity is single valued)

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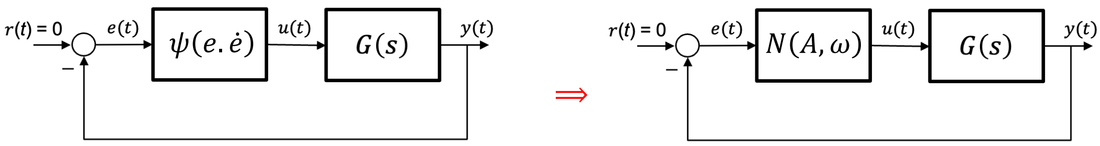
2. Describing function fundamentals

$$G(j\omega) = -\frac{1}{N(A,\omega)}$$



Characteristic equation: $1 + N(A,\omega)G(j\omega) = 0$

$$N(A,\omega) = \frac{j}{\pi A}\int_0^{2\pi} u(A\sin\psi, A\omega\cos\psi)\, e^{-j\psi}\, d\psi = n_p(A,\omega) + j n_q(A,\omega)$$

$$n_p(A,\omega) = \frac{1}{\pi A}\int_0^{2\pi} u(A\sin\psi, A\omega\cos\psi)\sin\psi\, d\psi$$

$$n_q(A,\omega) = \frac{1}{\pi A}\int_0^{2\pi} u(A\sin\psi, A\omega\cos\psi)\cos\psi\, d\psi$$

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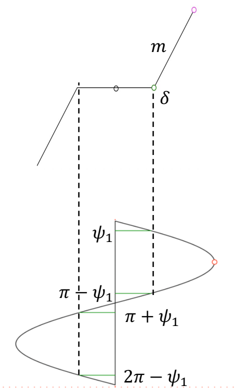
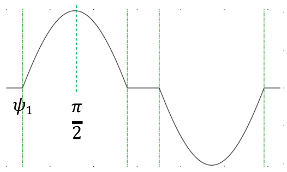
2. Describing function fundamentals

Simple examples of DF calculation

dead-zone





$\psi_1 = \sin^{-1}\left(\frac{\delta}{A}\right)$

$\sin\psi_1 = \frac{\delta}{A}$

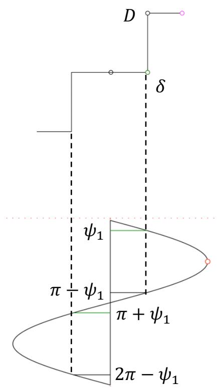
$\cos\psi_1 = \sqrt{1 - \frac{\delta^2}{A^2}}$

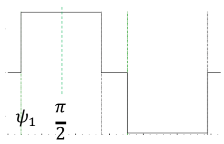
$N(A) = \frac{4}{\pi A}\int_{\psi_1}^{\pi/2}[m(A\sin\psi - \delta)]\sin\psi d\psi$

$= m\left[1 - f\left(\frac{\delta}{A}\right)\right]$

$f(\gamma) = \begin{cases} -1 & \gamma < -1 \\ \frac{2}{\pi}\left(\sin^{-1}\gamma + \gamma\sqrt{1-\gamma^2}\right) & |\gamma| \leqq 1 \\ 1 & \gamma > -1 \end{cases}$

ideal relay





$\psi_1 = \sin^{-1}\left(\frac{\delta}{A}\right)$

$\sin\psi_1 = \frac{\delta}{A}$

$\cos\psi_1 = \sqrt{1 - \frac{\delta^2}{A^2}}$

$N(A) = \frac{4}{\pi A}\int_{\psi_1}^{\pi/2} D\sin\psi d\psi = \frac{4D}{\pi A}\cos\psi_1$

$= \frac{2D}{A}q\left(\frac{\delta}{A}\right)$

$q(\gamma) = \begin{cases} \frac{2}{\pi}\sqrt{1-\gamma^2} & |\gamma| \leqq 1 \\ 0 & |\gamma| > 1 \end{cases}$

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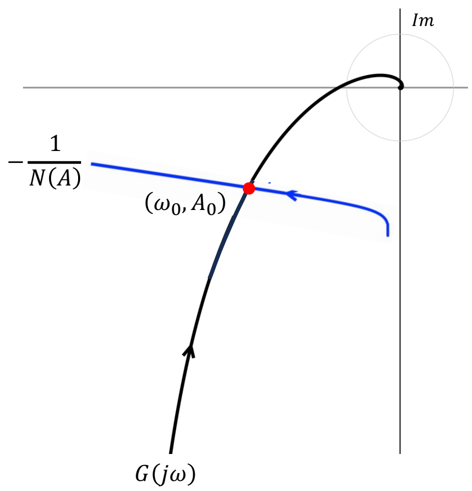
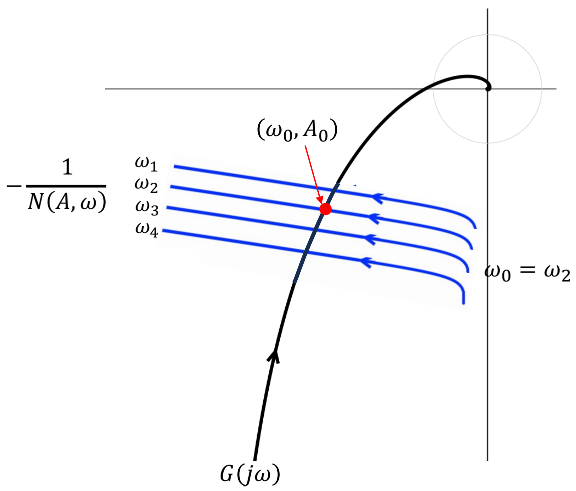
2. Describing function fundamentals

Graphical determination of limit cycles





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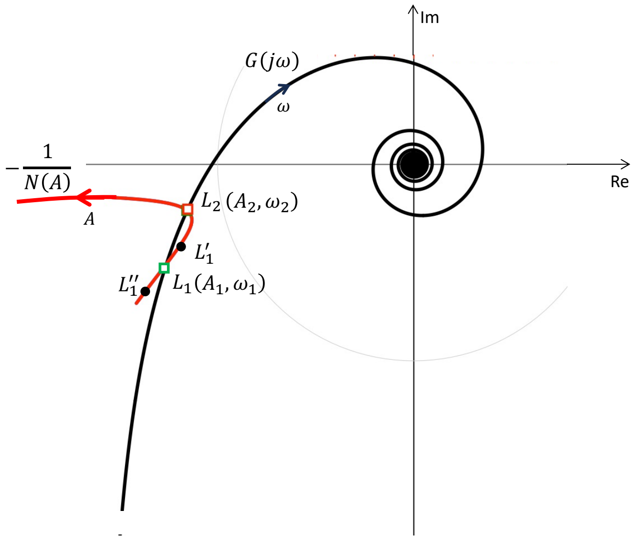
2. Describing function fundamentals

Limit cycle stability



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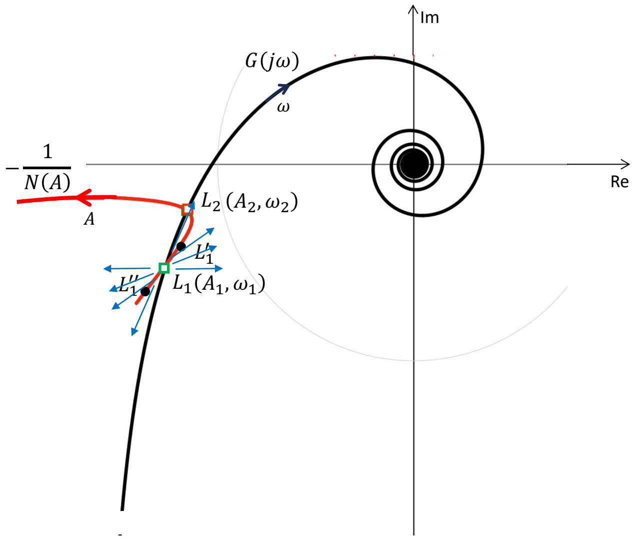
2. Describing function fundamentals

Limit cycle stability



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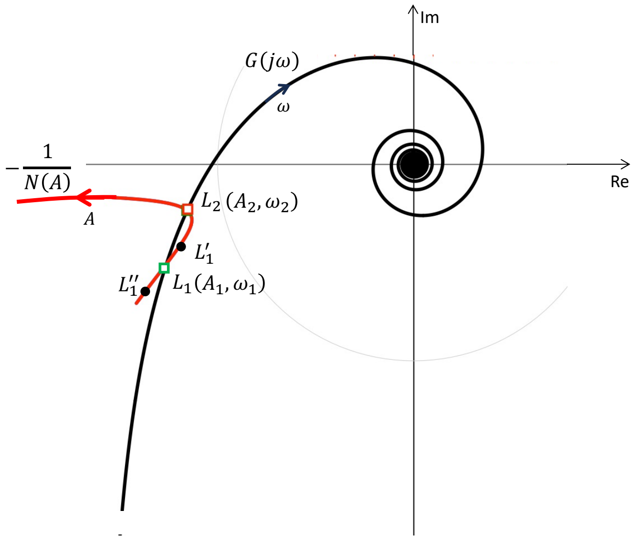
2. Describing function fundamentals

Limit cycle stability



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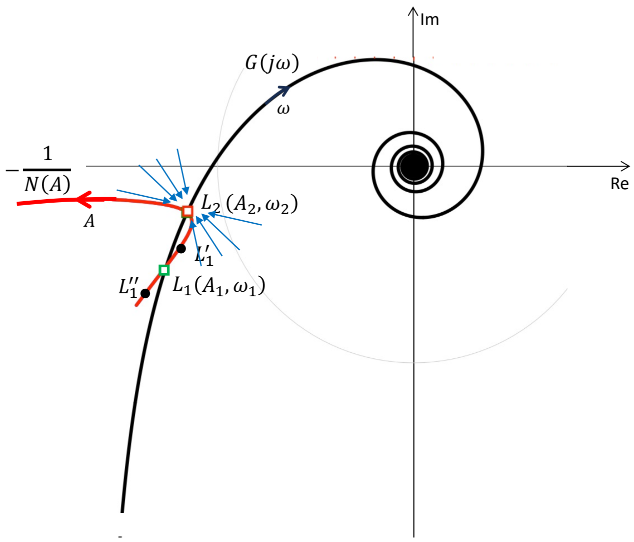
2. Describing function fundamentals

Limit cycle stability



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2. Describing function fundamentals

Limit cycle stability: Loeb criterion

$x(t) = Ae^{j\omega t} \longrightarrow x(t) = (A+\Delta A)e^{j(\omega+\Delta\omega+j\delta)t}$ (δ is the damping)

$$X(A,\omega) + jY(A,\omega) = 0 \qquad X(A+\Delta A, \omega+\Delta\omega+j\delta) + jY(A+\Delta A, \omega+\Delta\omega+j\delta) = 0$$

$$\begin{aligned} \frac{\partial X}{\partial A}\Delta A + \frac{\partial X}{\partial \omega}\Delta\omega - \frac{\partial Y}{\partial \omega}\delta = 0 \\ \frac{\partial Y}{\partial A}\Delta A + \frac{\partial Y}{\partial \omega}\Delta\omega + \frac{\partial X}{\partial \omega}\delta = 0 \end{aligned} \longrightarrow \left[\left(\frac{\partial X}{\partial \omega}\right)^2 + \left(\frac{\partial Y}{\partial \omega}\right)^2\right]\delta = \left[\frac{\partial X}{\partial A}\frac{\partial Y}{\partial \omega} - \frac{\partial Y}{\partial A}\frac{\partial X}{\partial \omega}\right]\Delta A \Longrightarrow \frac{\partial X}{\partial \omega}\frac{\partial X}{\partial \omega} - \frac{\partial X}{\partial \omega}\frac{\partial X}{\partial \omega} > 0$$

for the limit cycle to be stable it is necessary that: $sign(\delta) = sign(\Delta A)$

$$1 + N(A)L(j\omega) = 0$$

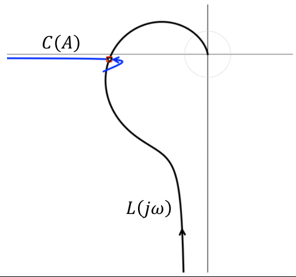
$$L = U(\omega) + jV(\omega)$$

$$C(A) = -\frac{1}{N(A)} = P(A) + jQ(A)$$

$$X(A,\omega) = U(\omega) - P(A) \longrightarrow \frac{\partial U}{\partial \omega}\frac{\partial Q}{\partial A} - \frac{\partial V}{\partial \omega}\frac{\partial P}{\partial A} > 0$$

$$Y(A,\omega) = V(\omega) - Q(A)$$

$$\frac{dL(j\omega)}{d\omega} \wedge \frac{dC(A)}{dA} > 0$$



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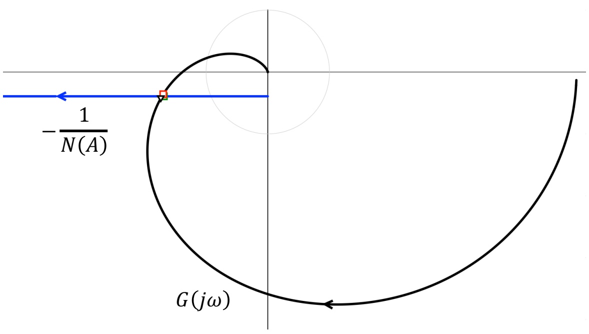
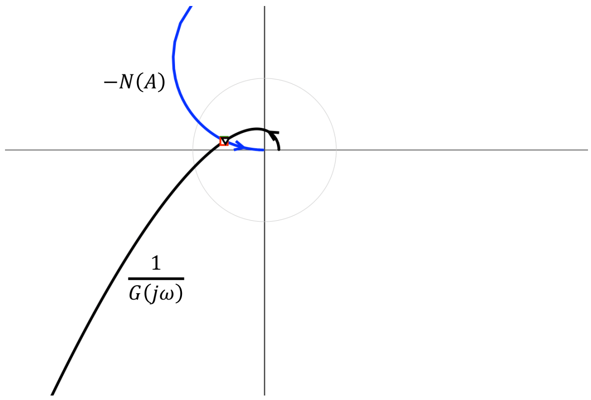
2. Describing function fundamentals

Alternative representations in the Nyquist diagram

$$G(j\omega) = -\frac{1}{N(A)}$$



$$-N(A) = \frac{1}{G(j\omega)}$$



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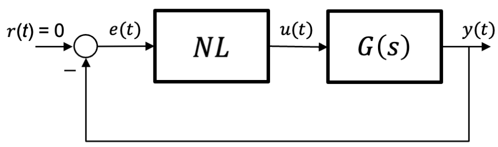
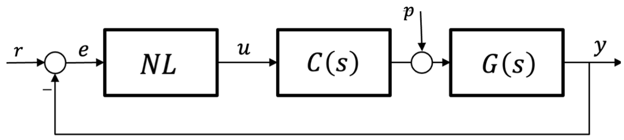
3. Dual Input Describing Function (DIDF)





$e(t) = A \sin \omega t \quad \Rightarrow \quad N_A(A)$

$e(t) = B + A \sin \omega t \quad \Rightarrow \quad N_A(A, B)$

$T\left|\frac{dr(t)}{dt}\right| \ll A$ $\quad$ *T* is the period of the limit cycle

$$G(j\omega) = -\frac{1}{N(A)}$$

$$G(j\omega) = -\frac{1}{N_A(A,B)}$$

From a practical point of view it is obvious that the value of *B* (the bias) is the average value of the signal at the input of the nonlinearity, i.e. $\Rightarrow \; B = \overline{e(t)}$

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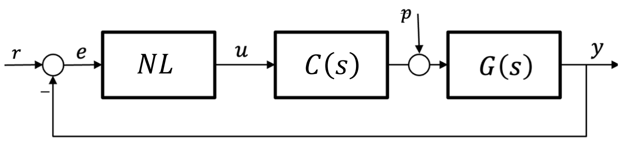
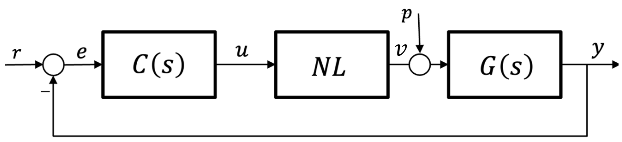
3. Dual Input Describing Function (DIDF)

Configuration $NL - C(s) - G(s)$



Configuration $C(s) - NL - G(s)$



The approximating gain to the bias input component *B* is given by:

$$N_B(A,B) = \frac{\overline{u(t)}}{\overline{e(t)}}$$

$$N_B(A,B) = \frac{\overline{v(t)}}{\overline{u(t)}}$$

It is important to understand how the bias *B* is transferred along the closed loop system and the influence that external signals *r*(*t*) and *d*(*t*) have on its value.

Without loss of generality, it will be assumed that the signals *r* and *d* are of the step type and that there is an oscillation with bias *B*.

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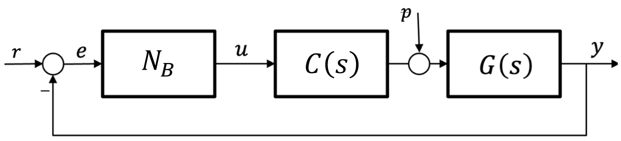
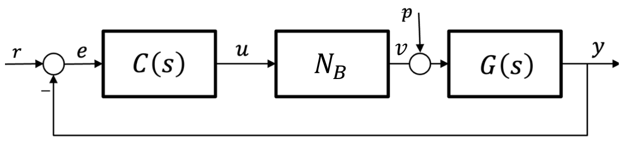
3. Dual Input Describing Function (DIDF)

Configuration $NL - C(s) - G(s)$



Configuration $C(s) - NL - G(s)$



It is only considered the non-oscillatory parts of the elements in the loop:

$$B = r_0 - y(\infty) = r_0 - G(0)[d_0 + BN_BC(0)]$$

$$B = C(0)r_0 - C(0)G(0)[d_0 + BN_B]$$

$$B(1 + C(0)G(0)N_B) = r_0 - G(0)d_0$$

$$B\left(1 + N_BC(0)G(0)\right) = C(0)r_0 - C(0)G(0)d_0$$

Example 1: $C(s)$ and $G(s)$ are TF of type 0

$$B\left(1 + N_BKK_p\right) = r_0 - K_pd_0$$

$$\left(1 + N_BKK_p\right) = K_p(r_0 - Kd_0)$$

Example 1: $C(s)$ and $G(s)$ are TF of type 1

$$B = 0$$

$$BN_B = -d_0$$

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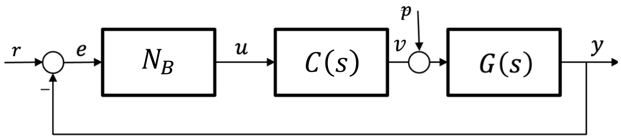
3. Dual Input Describing Function (DIDF)



Expanded requirements for any combination of process and controller for the maintenance of the bias under step-like changes in reference and disturbance inputs. $L_1(s)$ and $L_2(s)$ have neither poles nor zeros at the origin and it is considered that $n, m \geq 1$

Case	Structure of $C(s)$ and $G(s)$			Conditions
1	$C(s) = L_1(s)s^n$	$G(s) = L_2(s)s^m$		$B = r(\infty)$
2		$G(s) = L_2(s)$		$B = r(\infty) - p(\infty)L_2(0)$
3		$G(s) = L_2(s)/s^m$	$n > m$	$p(\infty) = 0$ & $B = r(\infty)$
4			$n = m$	$p(\infty) = 0$ & $B(1 + N_BL_1(0)L_2(0)) = r(\infty)$
5			$n < m$	$p(\infty) = 0$ & $B = 0$
6	$C(s) = L_1(s)$	$G(s) = L_2(s)s^m$		$B = r(\infty)$
7		$G(s) = L_2(s)$		$B(1 + N_BL_1(0)L_2(0)) = r(\infty) - p(\infty)L_2(0)$
8		$G(s) = L_2(s)/s^m$		$BN_B = -p(\infty)/L_1(0)$
9	$C(s) = L_1(s)/s^n$	$G(s) = L_2(s)/s^m$		$B = 0$
10		$G(s) = L_2(s)$		$B = 0$
11		$G(s) = L_2(s)s^m$	$n < m$	$B = r(\infty)$
12			$n = m$	$B(1 + N_BL_1(0)L_2(0)) = r(\infty)$
13			$n > m$	$B = 0$

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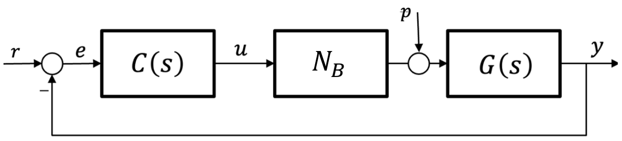
3. Dual Input Describing Function (DIDF)



Expanded requirements for any combination of process and controller for the maintenance of the bias under step-like changes in reference and disturbance inputs. $L_1(s)$ and $L_2(s)$ have neither poles nor zeros at the origin and it is considered that $n, m \geq 1$

Case	Structure of $C(s)$ and $G(s)$			Conditions
1	$C(s) = L_1(s)s^n$	$G(s) = L_2(s)s^m$		$B = 0$
2		$G(s) = L_2(s)$		$B = 0$
3		$G(s) = L_2(s)/s^m$	$n > m$	$B = 0$
4			$n = m$	$B(1 + \mathcal{N}_B L_1(0) L_2(0)) = -p(\infty) L_1(0) L_2(0)$
5			$n < m$	$B\mathcal{N}_B = -p(\infty)$
6	$C(s) = L_1(s)$	$G(s) = L_2(s)s^m$		$B = r(\infty) L_1(0)$
7		$G(s) = L_2(s)$		$B(1 + \mathcal{N}_B L_1(0) L_2(0)) = r(\infty) L_1(0) - p(\infty) L_1(0) L_2(0)$
8		$G(s) = L_2(s)/s^m$		$B\mathcal{N}_B = -p(\infty)$
9	$C(s) = L_1(s)/s^n$	$G(s) = L_2(s)/s^m$		$B\mathcal{N}_B = -p(\infty)$
10		$G(s) = L_2(s)$		$B\mathcal{N}_B = -r(\infty)/L_2(0) - p(\infty)$
11		$G(s) = L_2(s)s^m$	$n < m$	$r(\infty) = 0$ & $B = 0$
12			$n = m$	$r(\infty) = 0$ & $B(1 + \mathcal{N}_B L_1(0) L_2(0)) = -p(\infty) L_1(0) L_2(0)$
13			$n > m$	$r(\infty) = 0$ & $B\mathcal{N}_B = -p(\infty)$

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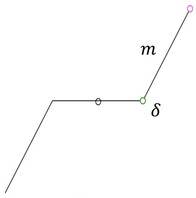
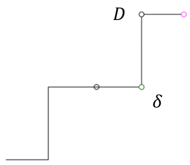
3. Dual Input Describing Function (DIDF)

DF **DIDF**



$$N(A) = m\left[1 - f\left(\frac{\delta}{A}\right)\right] \longrightarrow N_A(A, B) = m - 0.5m\left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right)\right]$$



$$N(A) = \frac{2D}{A} q\left(\frac{\delta}{A}\right) \longrightarrow N_A(A, B) = \frac{D}{A}\left[q\left(\frac{\delta + B}{A}\right) + q\left(\frac{\delta - B}{A}\right)\right]$$

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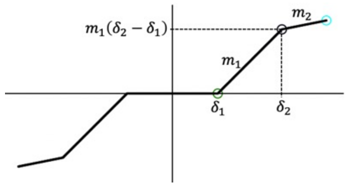
3. Dual Input Describing Function (DIDF)

When there are external signals acting on the control loop, it may happen that the nonlinearity seen by the process is different from the existing nonlinearity.

Example: Dead-zone + saturation



$$A_{min} = \delta_2 + |B|$$

$$n_p = 0.5(m_1 - m_2)\left[f\left(\frac{\delta_2 - B}{A}\right) + f\left(\frac{\delta_2 + B}{A}\right)\right] - 0.5m_1\left[f\left(\frac{\delta_1 - B}{A}\right) + f\left(\frac{\delta_1 + B}{A}\right)\right] + m_2$$

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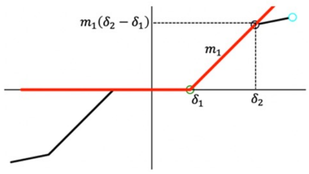
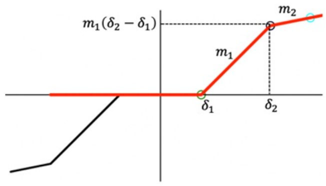
3. Dual Input Describing Function (DIDF)

When there are external signals acting on the control loop, it may happen that the nonlinearity seen by the process is different from the existing nonlinearity.



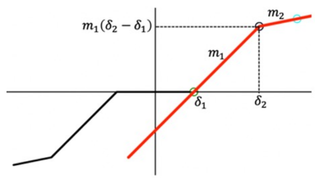
$A_{min} = \delta_1 - B,$

$n_p = 0.5m_1\left(1 - f\left(\frac{\delta_1 - B}{A}\right)\right)$



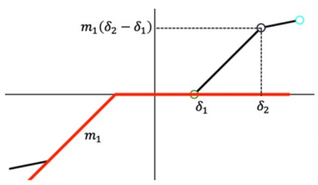
$A_{min} = \delta_2 - B,$

$n_p = 0.5(m_1 - m_2)f\left(\frac{\delta_2 - B}{A}\right) - 0.5\left(m_1 f\left(\frac{\delta_1 - B}{A}\right) - m_2\right)$



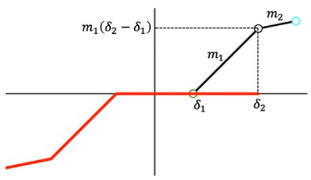
$A_{min} = \delta_2 - B,$

$n_p = 0.5(m_1 - m_2)f\left(\frac{\delta_2 - B}{A}\right) + 0.5(m_1 + m_2)$



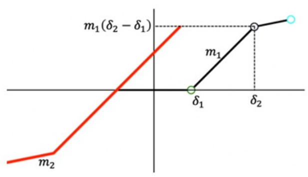
$A_{min} = \delta_1 + B,$

$n_p = 0.5m_1\left(1 - f\left(\frac{\delta_1 + B}{A}\right)\right)$



$A_{min} = \delta_2 + B,$

$n_p = 0.5(m_1 - m_2)f\left(\frac{\delta_2 + B}{A}\right) - 0.5\left(m_1 f\left(\frac{\delta_1 + B}{A}\right) - m_2\right)$



$A_{min} = \delta_2 + B,$

$n_p = 0.5(m_1 - m_2)f\left(\frac{\delta_2 + B}{A}\right) + 0.5(m_1 + m_2)$

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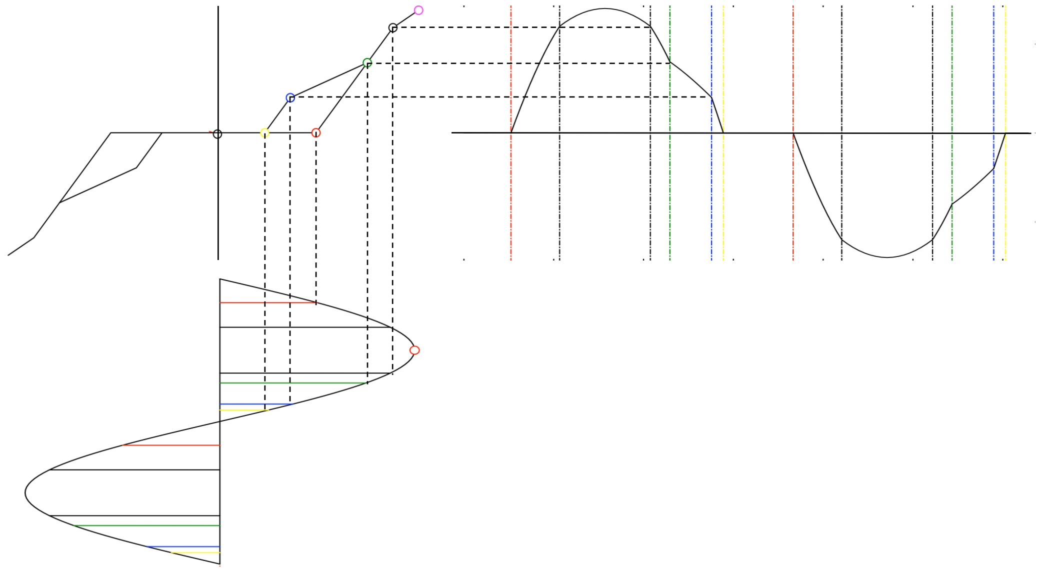
4. Describing function calculation



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4. Describing function calculation

The DIDF sinusoidal gain is given by:

$$N_A(A,B) = n_p(A,B) + jn_q(A,B) = \frac{j}{\pi A}\int_0^{2\pi} u(B + A\sin\psi)\, e^{-j\psi}\, d\psi =$$

$$n_p(A,B) = \frac{1}{\pi A}\int_0^{2\pi} u(B + A\sin\psi)\sin\psi\, d\psi \qquad n_q(A,B) = \frac{1}{\pi A}\int_0^{2\pi} u(B + A\sin\psi)\cos\psi\, d\psi$$

$n_q(A,B) = 0$ (if the nonlinearity is univaluated)

and the corresponding dc gain is given by:

$$N_B(A,B) = \frac{1}{2\pi B}\int_0^{2\pi} u(B + A\sin\psi) d\psi = \frac{\overline{u(t)}}{B}$$

1) Parallel composition of non-linear elements

2) Procedure for calculating $n_p(A,B)$ if the nonlinearity is single valued

3) Procedure for calculating $n_q(A,B)$ if the nonlinearity is multivalued

4) Given a multivalued nonlinearity, is there a single valued nonlinearity with the same value of $n_p(A,B)$?

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4. Describing function calculation

1) Parallel composition of non-linear elements:

Consider n nonlinearities N_1, N_2, …., N_n in parallel such that their total output $y(x)$ is:

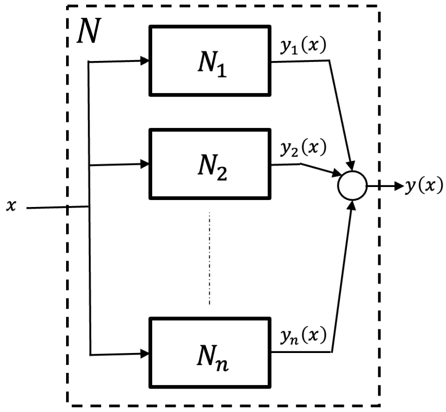
$$y(x) = y_1(x) + y_2(x) + \cdots + y_n(x) = \sum_{i=1}^{n} y_i(x)$$

It then follows that the DF for the composite nonlinearity N is given by:

$$N(A) = \frac{j}{\pi A}\int_0^{2\pi}\left[\sum_{i=1}^{n} y_i(A\sin\psi)\right]e^{-j\psi}d\psi$$

$$= \frac{j}{\pi A}\sum_{i=1}^{n}\int_0^{2\pi} y_i(A\sin\psi)\, e^{-j\psi}d\psi$$

$$= \sum_{i=1}^{n} N_i(A)$$

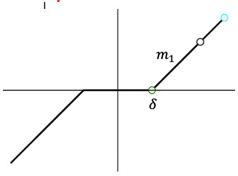


4. Describing function calculation

2) Procedure for calculating $n_p(A,B)$ if the nonlinearity is single valued

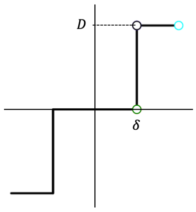
Any piecewise-linear odd memoryless nonlinearity can be composed using only two basic types of nonlinearities

Relay with dead zone



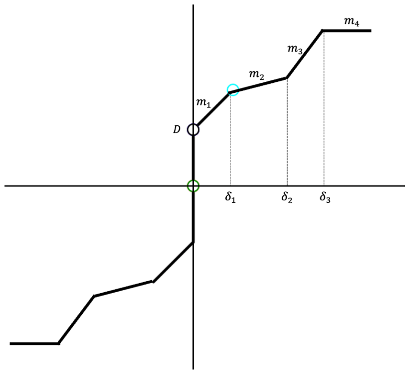
$$n_p = m_1\left\{1 - 0.5\left[f\left(\frac{\delta+B}{A}\right) + f\left(\frac{\delta-B}{A}\right)\right]\right\}$$

Dead zone



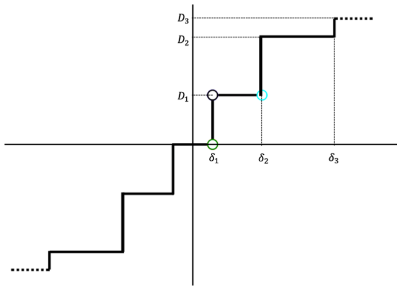
$$n_p = \frac{D}{A}\left[q\left(\frac{\delta+B}{A}\right) + q\left(\frac{\delta-B}{A}\right)\right]$$

Example 1



$$n_p = \frac{D}{A}\left[q\left(\frac{\delta+B}{A}\right) + q\left(\frac{\delta-B}{A}\right)\right] +$$
$$0.5\sum_{i=1}^{n}(m_i - m_{i+1})\left[f\left(\frac{\delta_i+B}{A}\right) + f\left(\frac{\delta_i-B}{A}\right)\right] + m_{n+1}$$

Example 2



$$n_p = \frac{1}{A}\sum_{i=1}^{n}(D_i - D_{i+1})\left[q\left(\frac{\delta_i+B}{A}\right) + q\left(\frac{\delta_i-B}{A}\right)\right]$$

4. Describing function calculation

3) Procedure for calculating $n_q(A, B)$ if the nonlinearity is multivalued

Let be a static, but otherwise, nonlinearity $u = u(x)$. Using the obvious relationship $d(B + A\sin\psi) = A\cos\psi\, d\psi$

it follows that $n_q(A, B)$ can be written as:

$$n_q(A,B) = \frac{1}{\pi A}\int_0^{2\pi} u(B + A\sin\psi)\cos\psi\, d\psi$$

$$= \frac{1}{\pi A^2}\oint u(B + A\sin\psi)d(B + A\sin\psi)$$

$$= \frac{1}{\pi A^2}\oint u(x)dx$$

$$= -\frac{S}{\pi A^2}$$

where S is the area enclosed by $u = u(x)$ as x varies through a complete cycle between $B + A$ and $B - A$

Gelb, A. and Van der Velde, W. E.: *Multiple-input describing functions and nonlinear system design*, McGraw-Hill. 1968

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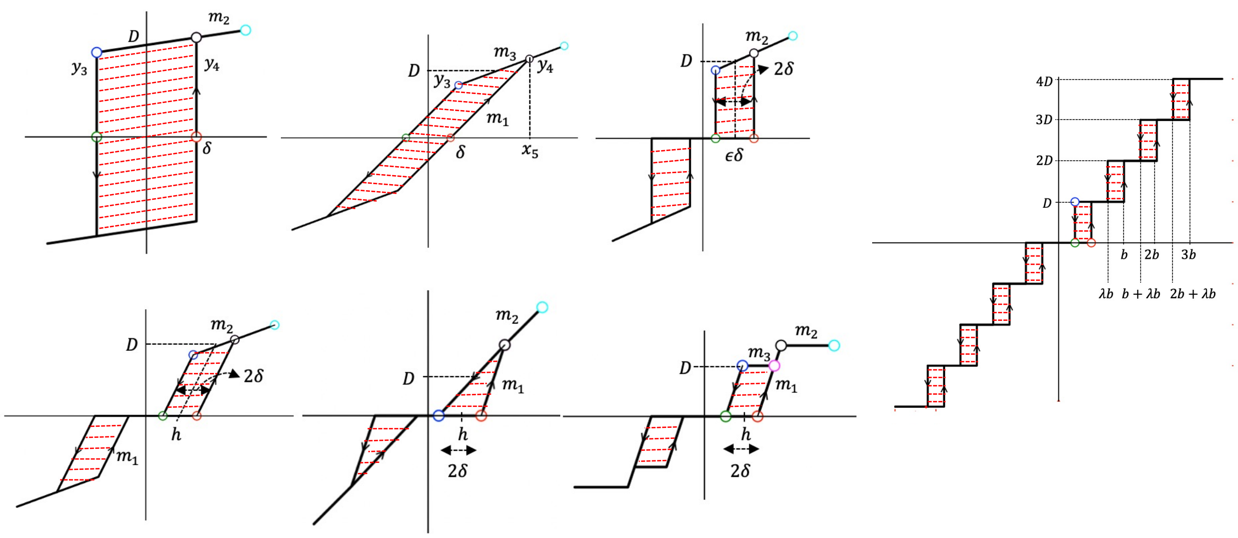
4. Describing function calculation

3) Procedure for calculating $n_q(A, B)$ if the nonlinearity is multivalued (some examples)



$$n_q(A,B) = -\frac{4D\delta}{\pi A^2}$$

$$n_q(A,B) = -\frac{8Db(1-\lambda)}{\pi A^2}$$

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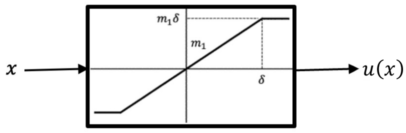
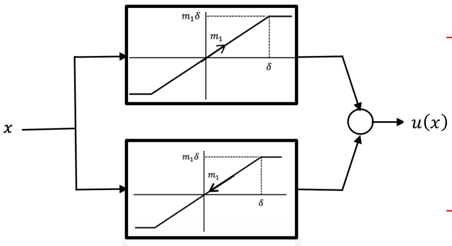
4. Describing function calculation

4) Given a multivalued nonlinearity, is there a single valued nonlinearity with the same value of $n_p(A, B)$?



$$n_p = 0.5 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$

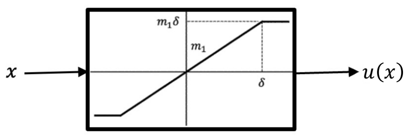


$$\rightarrow n_p = 0.25 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$

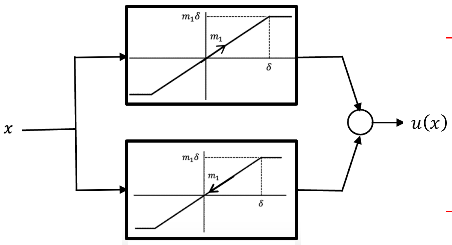
$$\rightarrow n_p = 0.25 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$

4. Describing function calculation

4) Given a multivalued nonlinearity, is there a univaluated nonlinearity with the same value of $n_p(A, B)$?

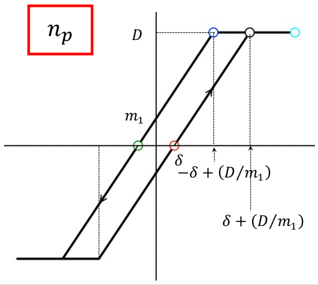


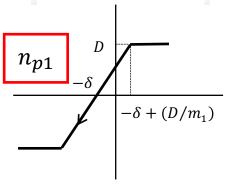
$$n_p = 0.5 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$

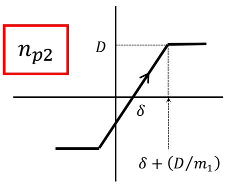


$$\rightarrow n_p = 0.25 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$

$$\rightarrow n_p = 0.25 m_1 \left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right) \right]$$







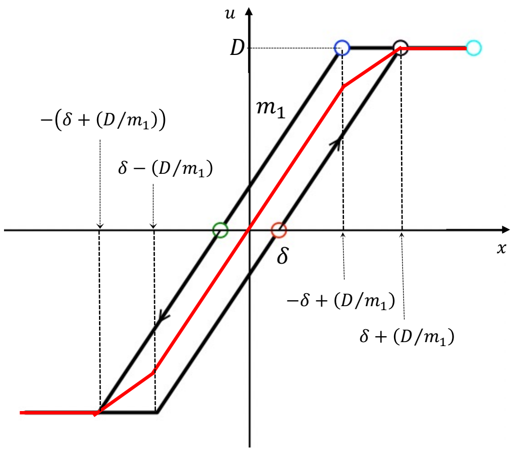
$$n_p = 0.5(n_{p1} + n_{p2})$$

$$\Downarrow$$

$$u(x) = 0.5(u_1(x) + u_2(x))$$

4. Describing function calculation

4) Given a multivalued nonlinearity, is there a univaluated nonlinearity with the same value of $n_p(A,B)$?



$$n_p = \frac{2}{\pi A}\int_0^{\pi} u(A\sin\varphi)\sin\varphi \, d\varphi, \qquad A \geq \delta + \frac{D}{m_1}$$

$$u = \begin{cases} D & x \geq \delta + (D/m_1) \\ 0.5 m_1 x + 0.5(D - \delta m_1) & -\delta + (D/m_1) < x < \delta + (D/m_1) \\ m_1 x & \delta - (D/m_1) \leq x \leq -\delta + (D/m_1) \\ 0.5 m_1 x - 0.5(D - \delta m_1) & -(\delta + (D/m_1)) < x < \delta - (D/m_1) \\ -D & x \leq -(\delta + (D/m_1)) \end{cases}$$

$$n_p = 0.25 m_1 \left[f\left(\frac{\frac{D}{m_1} + \delta + B}{A}\right) + f\left(\frac{\frac{D}{m_1} + \delta - B}{A}\right) + f\left(\frac{\frac{D}{m_1} - \delta + B}{A}\right) + f\left(\frac{\frac{D}{m_1} - \delta - B}{A}\right) \right]$$

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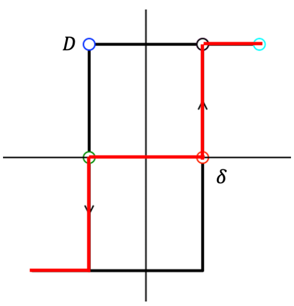
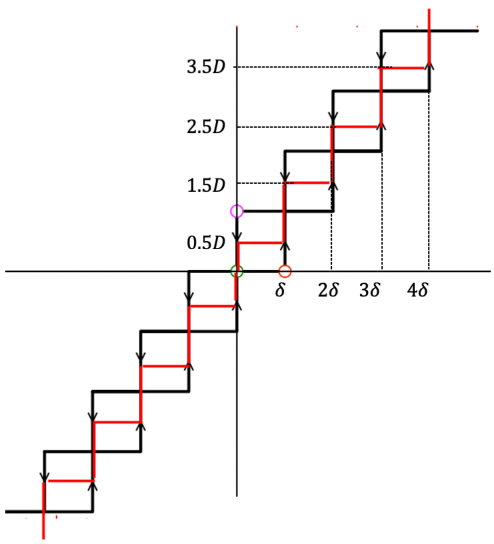
4. Describing function calculation

4) Given a multivalued nonlinearity, is there a univaluated nonlinearity with the same value of $n_p(A,B)$?



$$q(\gamma) = \begin{cases} \frac{2}{\pi}\sqrt{1-\gamma^2} & |\gamma| \leqq 1 \\ 0 & |\gamma| > 1 \end{cases}$$

$$n_p = \frac{D}{A}\left[q\left(\frac{\delta + B}{A}\right) + q\left(\frac{\delta - B}{A}\right) \right]$$



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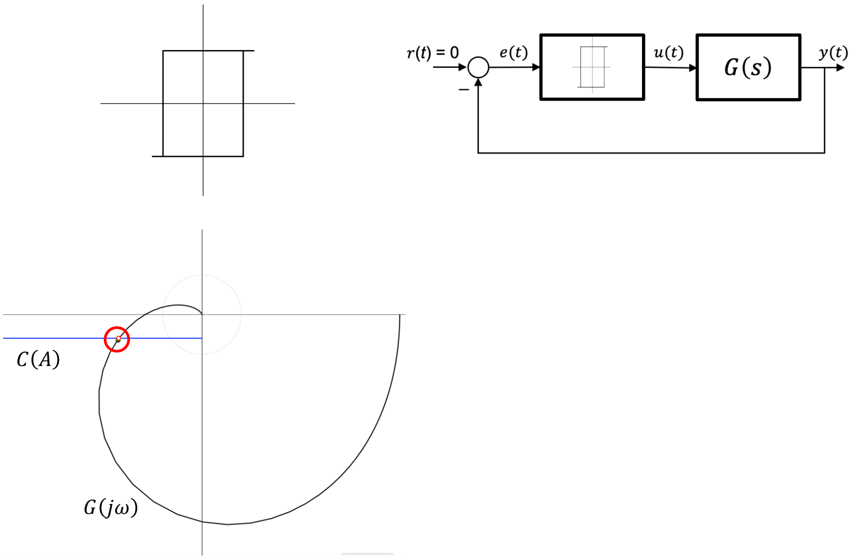
5. Example 1: The *n*-shifting method



K. J. Åström, T Hägglund.: Automatic tuning of simple regulators with specifications on phase and amplitude margins, *Automatica* 20, 5, 1984, 645-651

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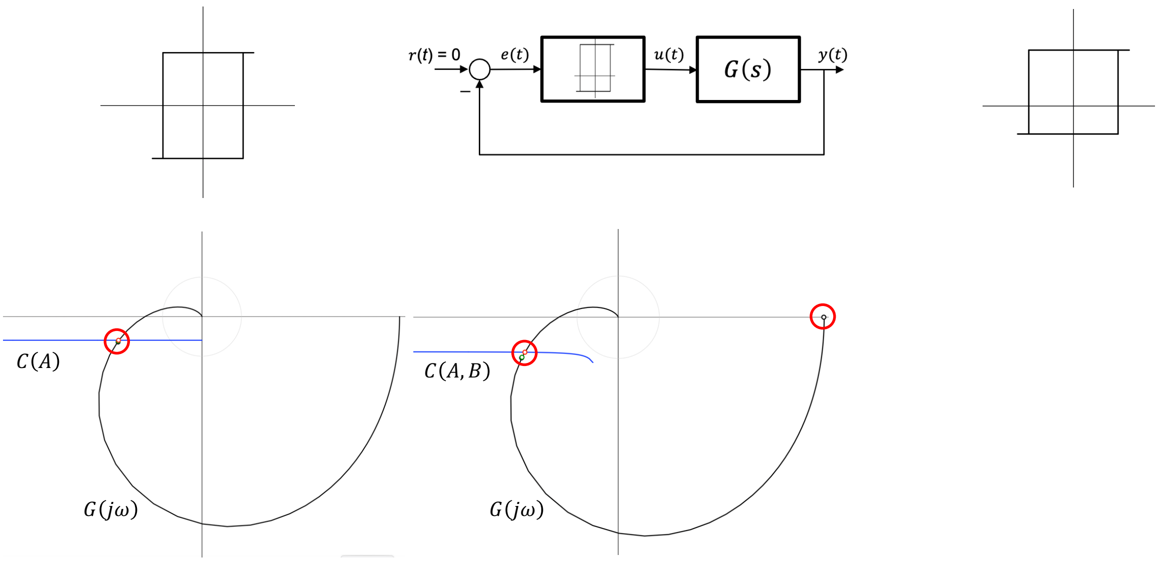
5. Example 1: The *n*-shifting method



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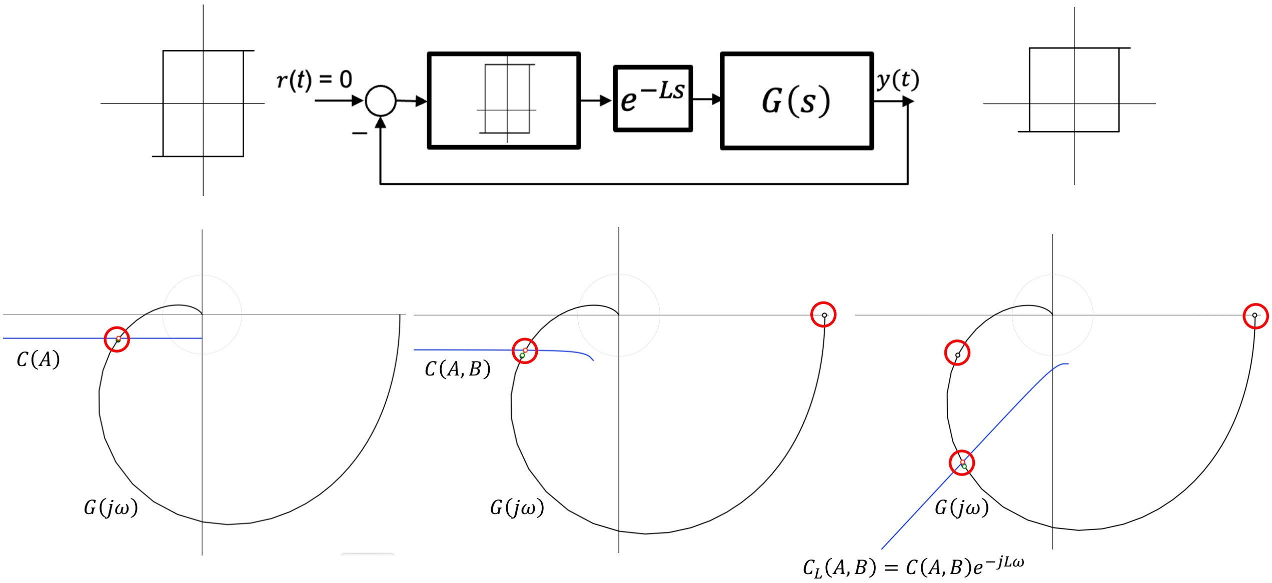
5. Example 1: The *n*-shifting method



K. J. Åström, T Hägglund.: Automatic tuning of simple regulators with specifications on phase and amplitude margins, *Automatica* 20, 5, 1984, 645-651

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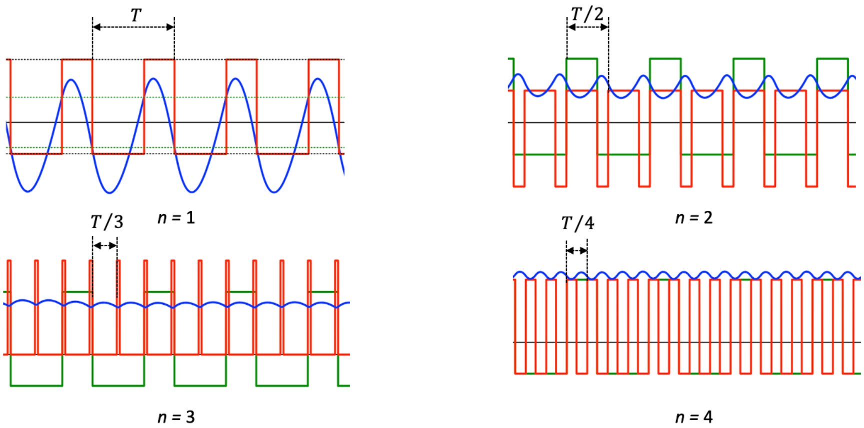
5. Example 1: The *n*-shifting method

Theorem: *Given a periodic function f(t) having a period T, that is, f(t) = f(t + T), then:*

$$\bar{f}(t) = \sum_{j=0}^{n-1} f\left(t - \frac{jT}{n}\right) = f(t) + f\left(t - \frac{T}{n}\right) + \cdots + f\left(t - \frac{(n-1)T}{n}\right)$$

is a periodic function of period T/n.



J. Sánchez, S. Dormido, J. M. Díaz. "Fitting of generic process models by an asymmetric short relay feedback experiment: The *n*-shifting method", *Appl. Sci.,* 2021, 11, 1651

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5. Example 1: The *n*-shifting method

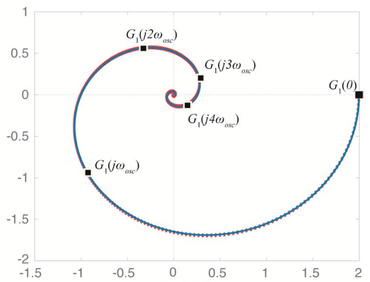
Theorem: *Given a periodic function f (t) having a period T, that is f (t) = f (t + T) then:*

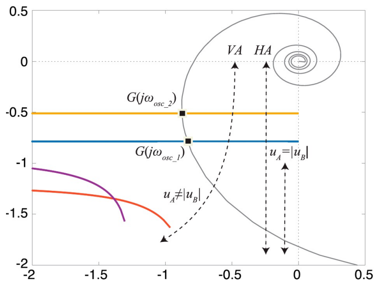
$$f(t) = \sum_{j=0}^{n-1} f\left(t - \frac{jT}{n}\right) = f(t) + f\left(t - \frac{T}{n}\right) + \cdots f\left(t - \frac{(n-1)T}{n}\right)$$ *is a periodic function of period T/n*

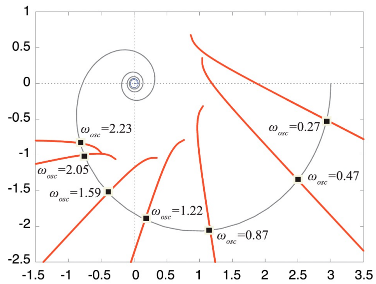
Proof: $f(t) \leftrightarrow F(s) \;\Rightarrow\; f\left(t - \frac{jT}{n}\right) \leftrightarrow e^{\frac{-jT}{n}} F(s)$ *then* $F(s) = \left(1 + e^{\frac{-Ts}{n}} + e^{\frac{-2Ts}{n}} \cdots + e^{\frac{-(n-1)Ts}{n}}\right) F(s)$

It is well known that the Laplace transform of a periodic function f (t) = f (t + T) is:

$$F(s) = \frac{\int_0^T e^{-sT} f(t)\, dt}{1 - e^{-sT}} = \frac{F_0(s)}{1 - e^{-sT}} \quad \Rightarrow \quad F(s) = \frac{\left(1 + e^{\frac{-Ts}{n}} + e^{\frac{-2Ts}{n}} \cdots + e^{\frac{-(n-1)Ts}{n}}\right)}{1 - e^{-sT}} F_0(s) = \frac{1}{1 - e^{\frac{-sT}{n}}} F_0(s)$$



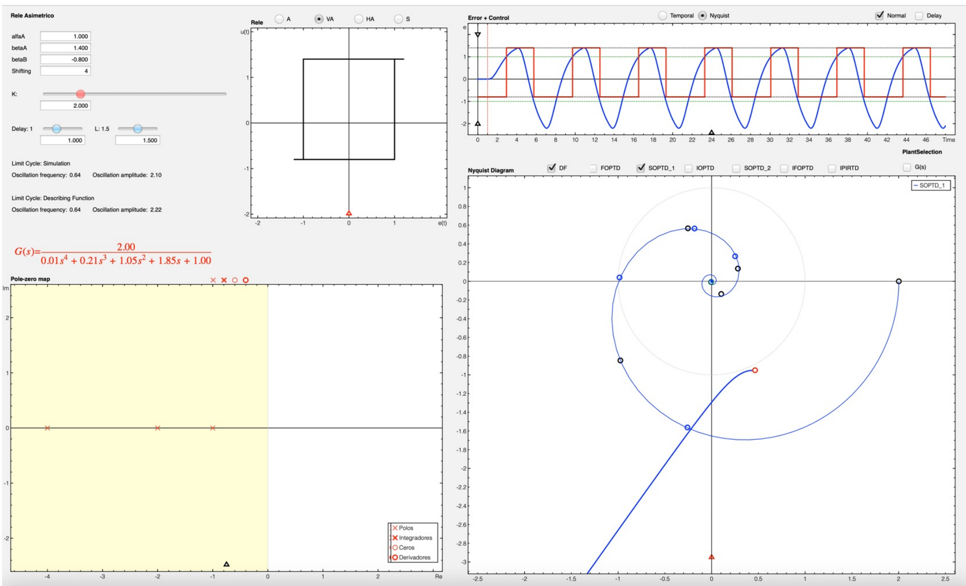




J. Sánchez, S. Dormido, O. Escrig, J. Ariel Romero. "Asymmetric delayed relay feedback identification based on the *n*-shifting approach", *Int. J. of Control*, 2024 , 97, 1, 59-71

53

5. Example 1: The *n*-shifting method



O. Escrig, J. A. Romero, J. Sánchez, S. Dormido. "Multiple frequency response points identification through single asymmetric relay feedback experiment", *Automatica*, 147, 2023, 110749, doi: 10.1016/j.automatica.2023.110749)

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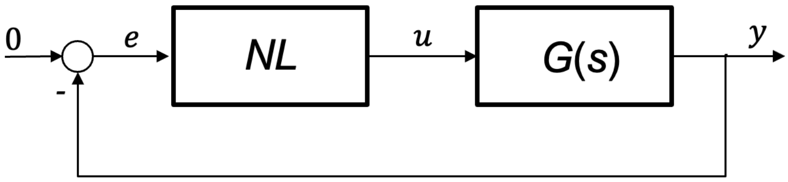
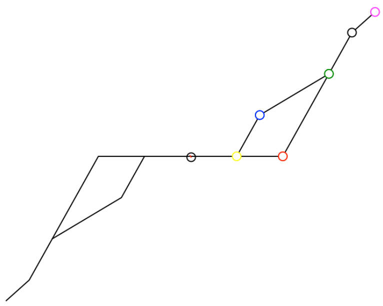
6. Example 2: The Shridar nonlinearity

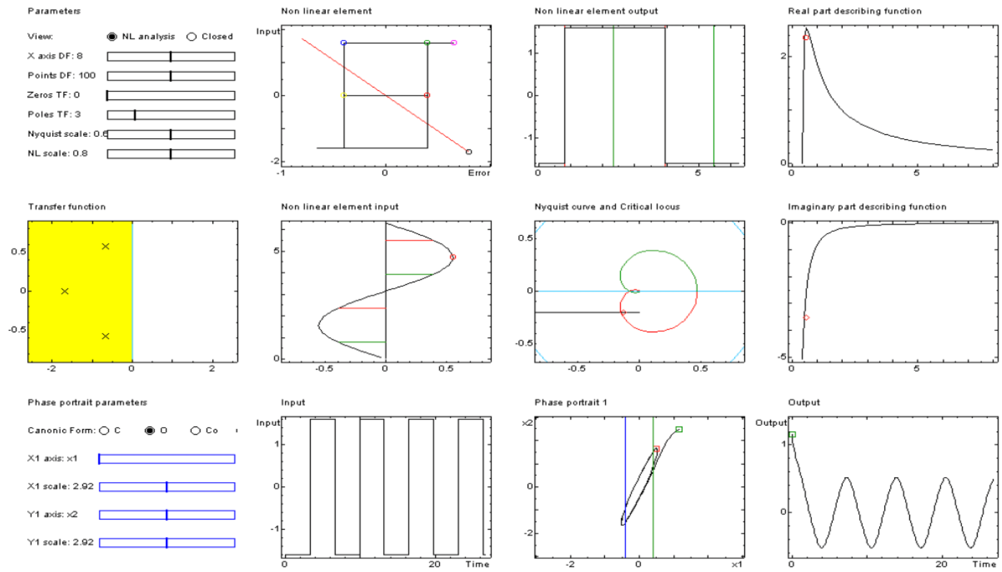




Dormido, S.; Gordillo, F.; Dormido Canto, S.; Aracil, J. "An interactive tool for introductory nonlinear control systems education", *IFAC World Congress b'02*, Barcelona, 2002

56

6. Example 2: The Shridar nonlinearity



Dormido, S.; Gordillo, F.; Aracil, J. "An interactive tool for introductory nonlinear control systems education", *IFAC World Congress b'02*, Barcelona, 2002

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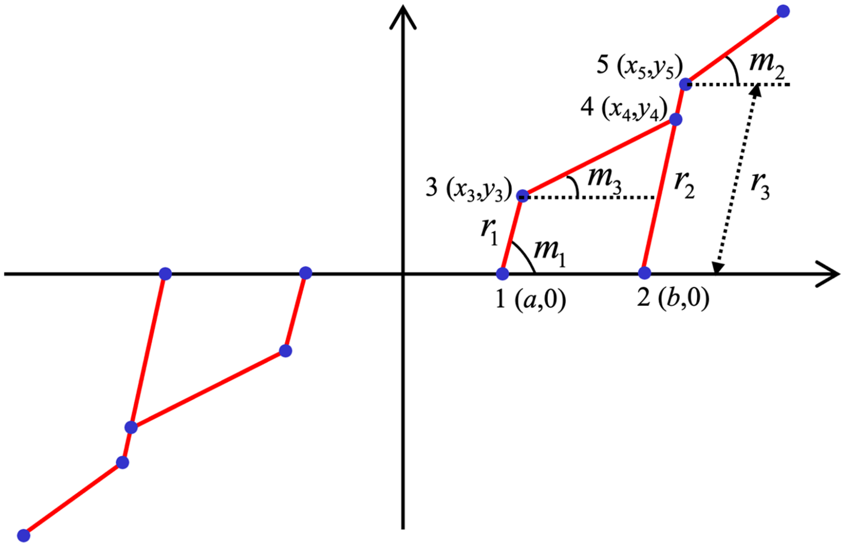
6. Example 2: The Shridar nonlinearity

Sridhar nonlinearity



Sridhar, R. "A general method for deriving the describing functions for a certain class of nonlinearities", *IRE Transactions on Automatic Control*, 5(2), 135–141,1960

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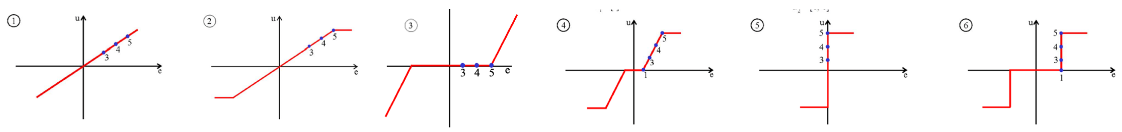
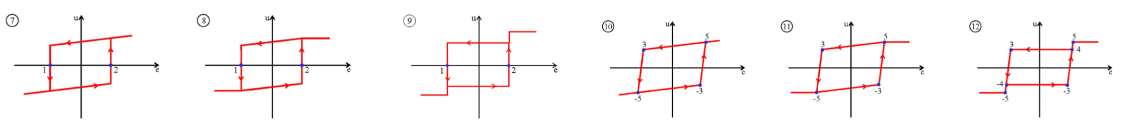
6. Example 2: The Shridar nonlinearity

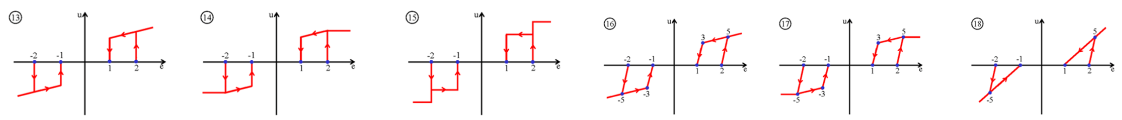
Single-valued nonlinearity (a = b)

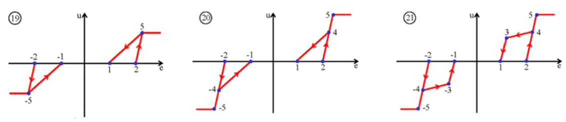


Nonlinearity with hysteresis (a = -b)



Nonlinearity with hysteresis + dead-zpne (-b < a < b)





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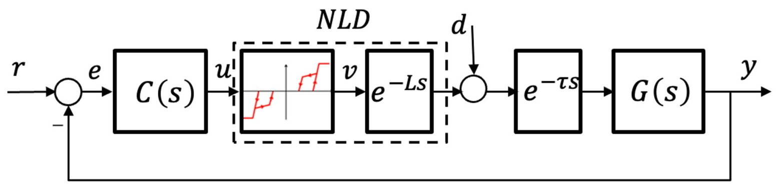
6. Example 2: The Shridar nonlinearity



Seven degrees of freedom

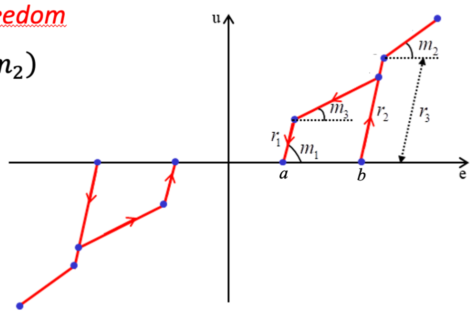
$(a, b, r_1, r_2, r_3, m_1, m_2)$

Constraints

$-b \leq a \leq b$

$0 \leq r_1 \leq r_2 \leq r_3$

$0 \leq m_2 \leq m_1$



Describing Function approach

$$NL_D(A)G(j\omega) = -1$$

Event-based simulation

$$\dot{x}(t) = Ax(t) + Bu(t - L - \tau)$$
$$e(t) = -Cx(t)$$
$$u(t) = f\big(e(t), \dot{e}(t)\big)$$

S. Dormido, C. Lampón, J. M. Díaz, R. Costa-Castelló. "Describing Function and Event Based Analysis of a General Class of Piecewise Feedback Control Systems", *4th IFAC Conference on Advances in Proportional-Integral-Derivative Control*, (*PID 2024*), Almería, Spain, june, 364-369, 2024

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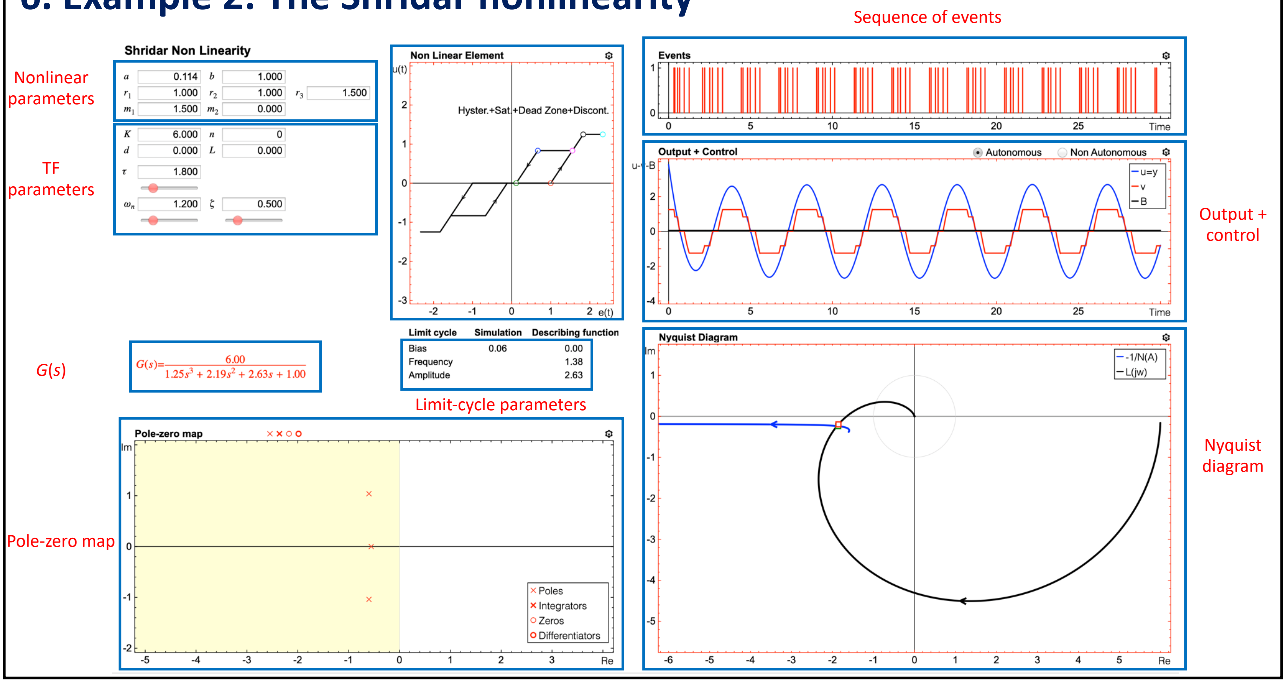
6. Example 2: The Shridar nonlinearity



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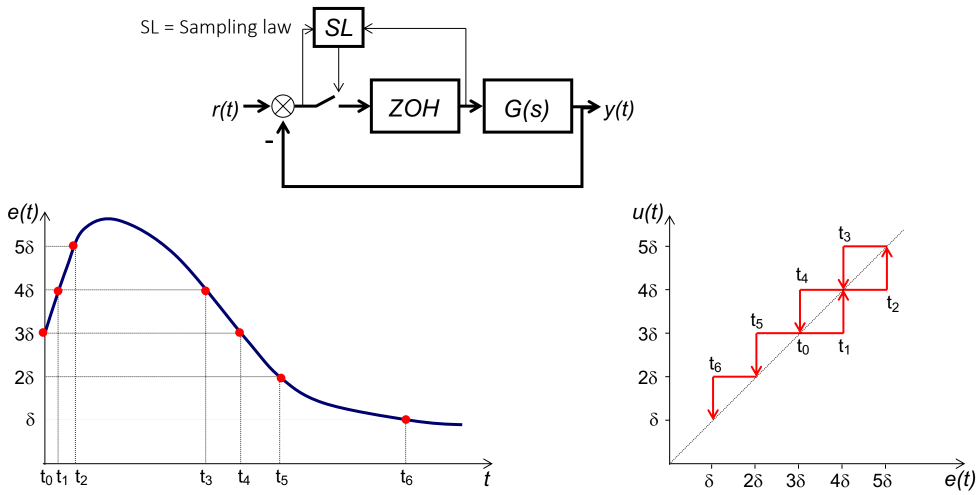
7. Example 3: Event-based sampling

An adaptive sampling law: $|e(t) - e(t_k)| = \delta$



S. Dormido, M. Mellado. "A study on fixed-difference sampling scheme", *Applications and Research in Information Systems and Sciences*, 1973, pp. 480-500

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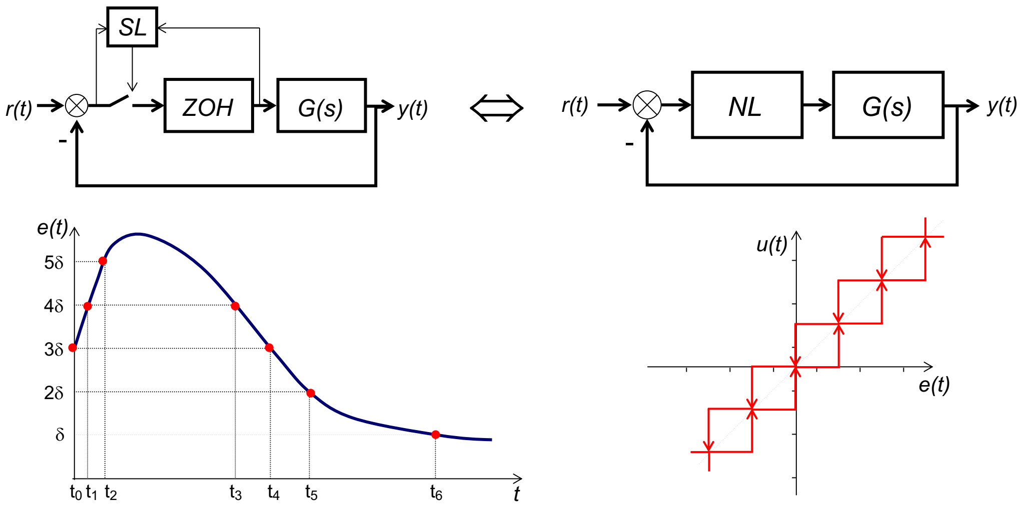
7. Example 3: Event-based sampling

An adaptive sampling law: $|e(t) - e(t_k)| = \delta$



S. Dormido, M. Mellado. "A study on fixed-difference sampling scheme", *Applications and Research in Information Systems and Sciences*, 1973, pp. 480-500

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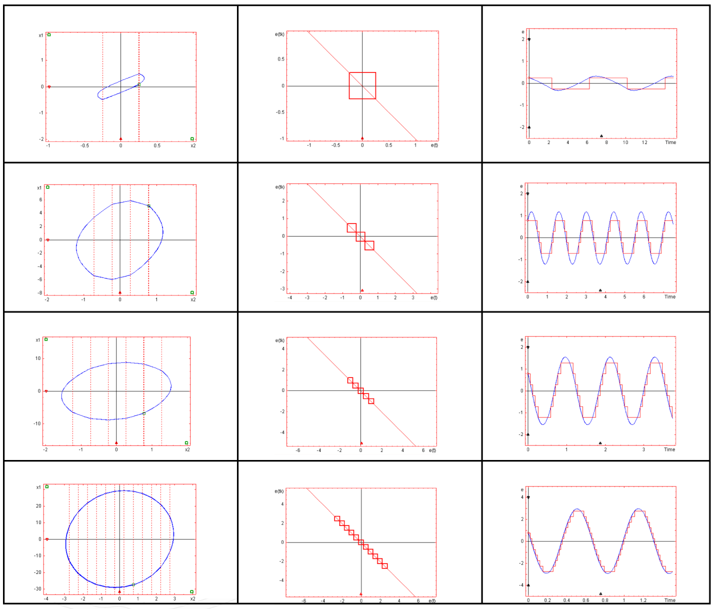
7. Example 3: Event-based sampling



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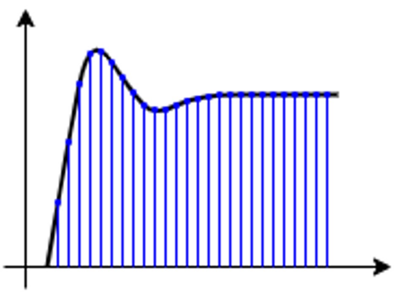
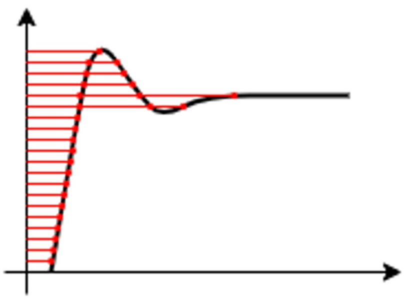
7. Example 3: Event-based sampling

Periodic sampling vs Send on delta sampling



Periodic sampling



Send-on delta sampling

K. J. Åström, B. M. Bernhardsson, Comparison of Riemann and Lebesgue sampling for first order stochastic systems, in: Proceedings of the 41st *IEEE Conference on Decision and Control*, 2002., Vol. 2, IEEE, 2002, pp. 2011–2016. doi:10.1109/CDC.2002.1184824.

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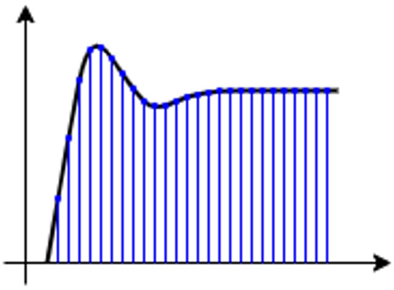
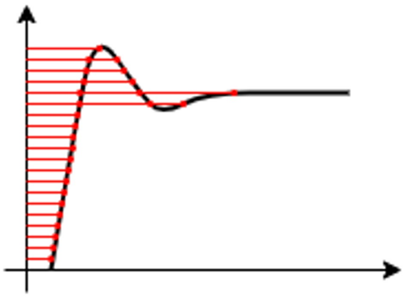
7. Example 3: Event-based sampling

Riemann vs Lebesgue sampling



Riemann sampling



Lebesgue sampling

K. J. Åström, B. M. Bernhardsson, Comparison of Riemann and Lebesgue sampling for first order stochastic systems, in: Proceedings of the 41st *IEEE Conference on Decision and Control*, 2002., Vol. 2, IEEE, 2002, pp. 2011–2016. doi:10.1109/CDC.2002.1184824.

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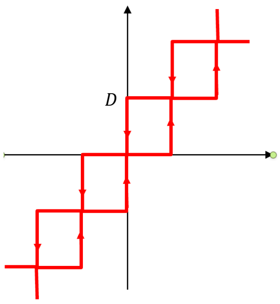
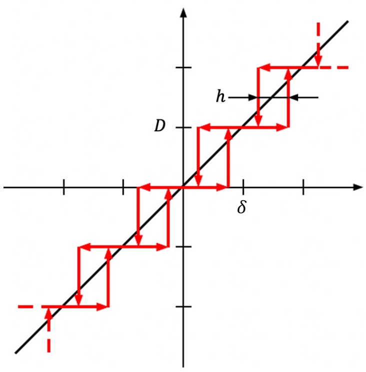
7. Example 3: Event-based sampling

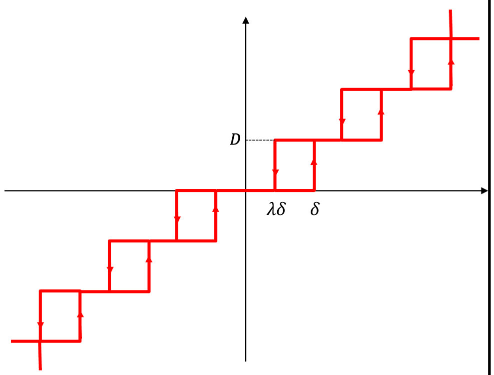
Event-based sampling



SSOD



RQH



RH+DZ

O. Escrig, J. A. Romero, J. Sánchez, S. Dormido. “Characterization of limit cycle oscillations induced by Fixed Threshold Samplers”, *IEEE Access*, 10, 2022, 62581-62596, doi: 10.1109/ACCESS.2022.3182794

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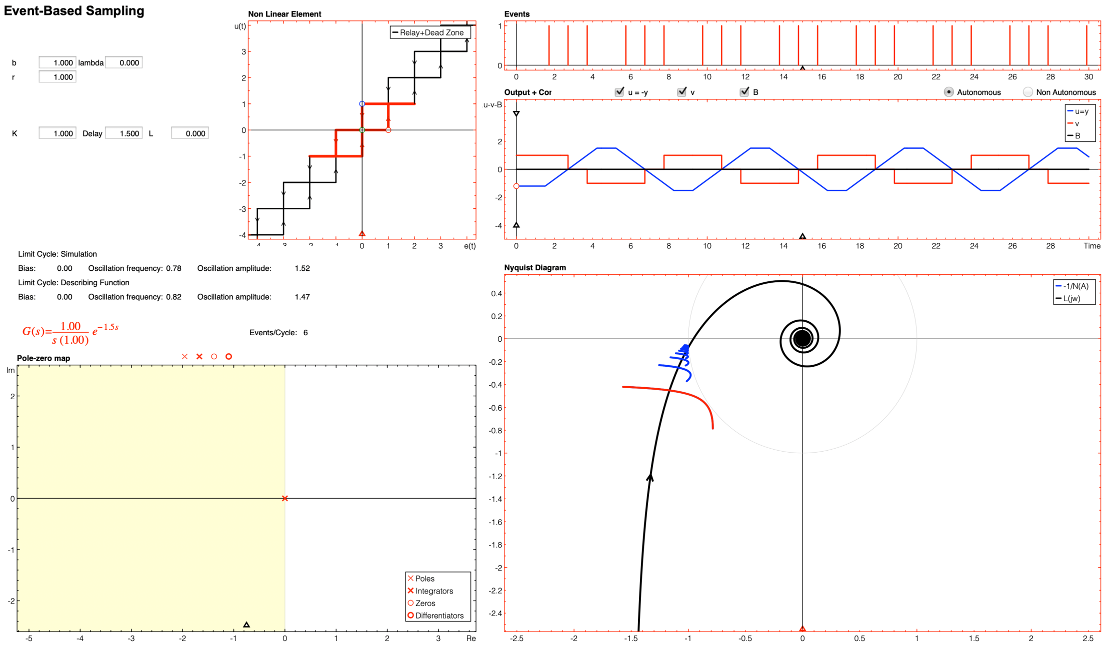
7. Example 3: Event-based sampling



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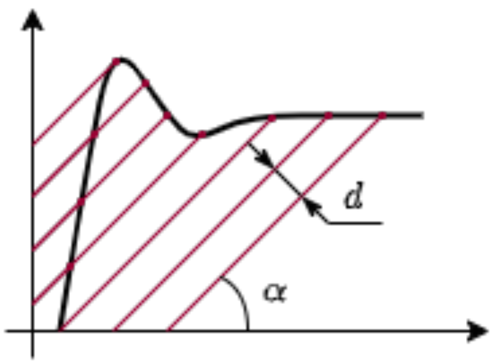
7. Example 3: Event-based sampling

Hybrid Riemann-Lebesgue sampling



Characteristic parameters:

- Distance between sampling lines *d*
- Slope α of the lines

O. Miguel-Escrig, J. A. Romero, J. Sánchez, S. Dormido. "Riemann-Lebesgue sampling for event-based control", *Automatica*, submitted (2025)

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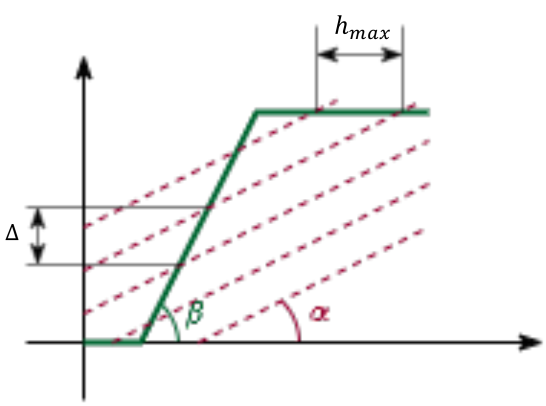
7. Example 3: Event-based sampling

Hybrid Riemann-Lebesgue sampling



Parameter selection

A certain waveform and maximum requirements are assumed:

- Quantification Δ
- Time between samples h_{max}

To obtain the parameters of the hybrid Riemann-Lebesgue sampler:

$$\frac{h_{max}}{\Delta} = \frac{1}{\tan\alpha} - \frac{1}{\tan\beta}$$
$$d = h_{max}\sin\alpha$$

O. Miguel-Escrig, J. A. Romero, J. Sánchez, S. Dormido. "Riemann-Lebesgue sampling for event-based control", *Automatica*, submitted (2025)

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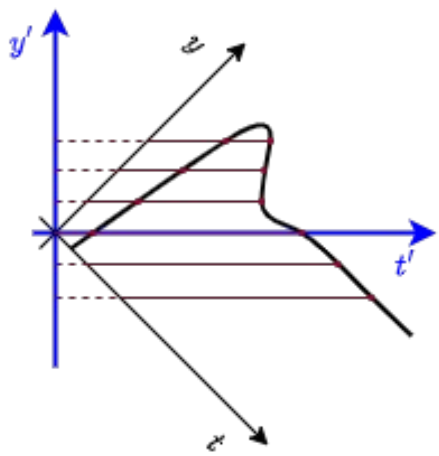
7. Example 3: Event-based sampling

Hybrid Riemann-Lebesgue sampling



Implementation

The sampled values can be rotated (-α rad) and sampled as a SOD, giving the following output as a function of the input:

$$\bar{y} = \begin{cases} y(t) & if \quad rem\left(\dfrac{y(t)\cdot\cos\alpha - t\cdot\sin\alpha}{d}\right) \\ \bar{y}(t^-) & otherwise \end{cases}$$

O. Miguel-Escrig, J. A. Romero, J. Sánchez, S. Dormido. "Riemann-Lebesgue sampling for event-based control", *Automatica*, submitted (2025)

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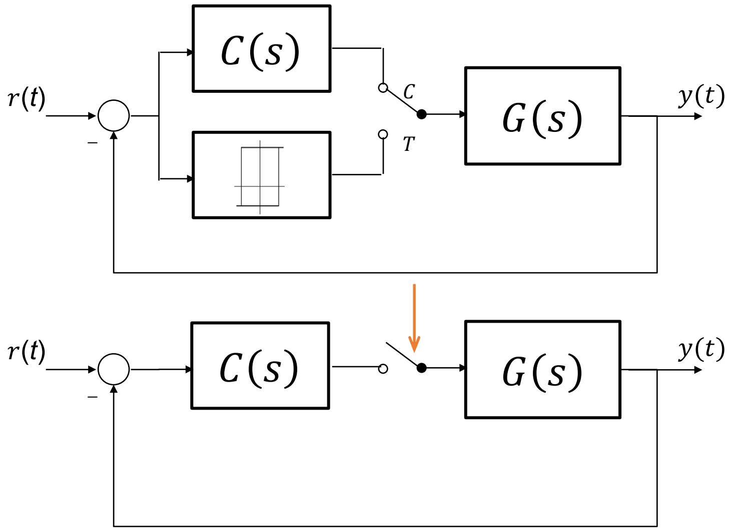
7. Example 3: Event-based sampling

Switching between Riemann and Lebesgue sampling



O. Miguel-Escrig, J. A. Romero, J. Sánchez, S. Dormido. "On-line Retuning of PID controllers with Fixed Threshold Samplers", *ISA Transactions*, 2023, doi:10.1016/j.isatra.2023.04.003

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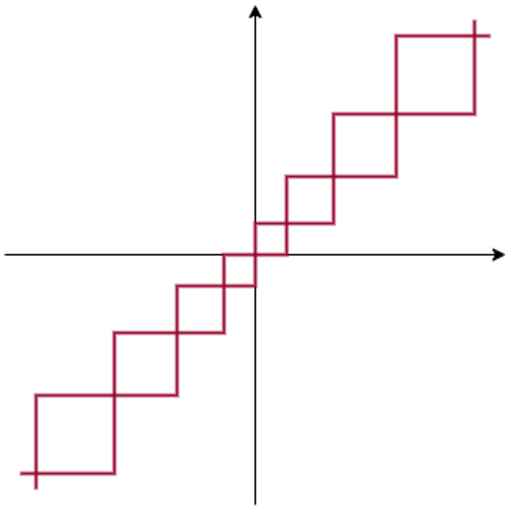
7. Example 3: Event-based sampling

SSOD type nonlinearities with non uniform thresholds



O. Miguel-Escrig, J. A. Romero, J. Sánchez, S. Dormido. "Error dependent sampling to reduce transient events in event based control", *IEEE Industrial Electronics Society*, ONCON 2024

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8. Conclusions

1. Importance of the study of nonlinear dynamic systems in the training of an automatic control engineer
2. It is surprising that its study is practically testimonial in many masters related to automatic control.
3. The Describing Function (DF) is a classical control technique that allows the determination of limit cycles
4. For the piecewise linear control systems DF allows to guarantee the existence of such limit cycles
5. Non-trivial examples have been presented that demonstrate its usefulness in the study of nonlinear control systems.

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Acknowledgment


José Manuel Diaz
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UNED


Julio Ariel Romero
UJI


Oscar Miguel Escrig
UJI

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